

# Existence of solutions and convergence of iterative algorithms for a system of generalized nonlinear mixed quasi-variational inclusions<sup>☆</sup>

Jian Wen Peng<sup>a,b,\*</sup>, Dao Li Zhu<sup>b</sup>

<sup>a</sup> College of Mathematics and Computer Science, Chongqing Normal University, Chongqing, 400047, PR China

<sup>b</sup> Department of Management Science, School of Management, Fudan University, Shanghai, 200433, PR China

Received 12 December 2005; received in revised form 29 June 2006; accepted 12 July 2006

---

## Abstract

In this paper, we introduce and study a new system of generalized nonlinear mixed quasi-variational inclusions in  $q$ -uniformly smooth Banach spaces. We prove the existence and uniqueness of solutions for this system of generalized nonlinear mixed quasi-variational inclusions. We also prove the convergence of several new two-step iterative algorithms with or without errors for this system of generalized nonlinear mixed quasi-variational inclusions. The results in this paper extend and improve some known results in the literature.

© 2007 Elsevier Ltd. All rights reserved.

**Keywords:** Two-step iterative algorithm; System of generalized nonlinear mixed quasi-variational inclusions; Strongly  $r$ -accretive mapping;  $\mu$ -Lipschitz continuous mapping; Existence;  $q$ -uniformly smooth Banach space

---

## 1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see [1–39] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [1], Cohen and Chaplais [2], Bianchi [3] and Ansari and Yao [4] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can all be modeled as a system of variational inequalities. Ansari et al. [5] considered a system of vector variational inequalities and obtained its existence results. Allevi et al. [6] considered a system of generalized vector variational inequalities and established

---

<sup>☆</sup> This paper was supported by the National Natural Science Foundation of China (Grant No. 70432001), the Science and Technology Research Project of Chinese Ministry of Education (Grant No. 206123) and the Postdoctoral Science Foundation of China (Grant No. 2005038133).

\* Corresponding author at: College of Mathematics and Computer Science, Chongqing Normal University, Chongqing, 400047, PR China.  
E-mail address: [jwpeng6@yahoo.com.cn](mailto:jwpeng6@yahoo.com.cn) (J.W. Peng).

some existence results with relative pseudomonotonicity. Kassay and Kolumbán [7] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay, Kolumbán and Páles [8] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani–Fan–Glicksberg fixed point theorem. Peng [9,10] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [11] introduced a two-step model for nonlinear variational inequalities and discussed the approximation solvability of this model based on the convergence analysis of a two-step projection method in Hilbert space setting. Nie et al. [12] investigated using this two-step model the approximation solvability of a system of nonlinear variational inequalities involving a combination of strongly monotone and pseudocontractive mappings. Verma [13] introduced a more general two-step model for projection method than the two-step model in [11] and then applied it to the approximation solvability of a two-step strongly monotone nonlinear variational inequality in a Hilbert spaces setting. Verma [14] investigated the approximation solvability of a new system of nonlinear quasi-variational inequalities in Hilbert spaces. Kim and Kim [15] introduced and studied a new system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. As generalizations of system of variational inequalities, Agarwal, Huang and Tan [16] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces.

Inspired and motivated by the results in [1–17], the purpose of this paper is to introduce and study a new system of generalized nonlinear mixed quasi-variational inclusions in real Banach spaces. By using the resolvent technique for the  $m$ -accretive mappings, we prove the existence and uniqueness of solutions for this system of generalized nonlinear mixed quasi-variational inclusions. We also prove the convergence of several new two-step iterative algorithms with or without errors to get an approximation of the solution for this system of generalized nonlinear mixed quasi-variational inclusions. The result in this paper extends and improves some results in [11–15,17] in several aspects.

## 2. Preliminaries

Throughout this paper we suppose that  $E$  is a real Banach space with dual space, norm and the generalized dual pair denoted by  $E^*$ ,  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively,  $2^E$  is the family of all the nonempty subsets of  $E$ ,  $\text{dom}(M)$  denotes the domain of the set-valued map  $M : E \rightarrow 2^E$  and the generalized duality mapping  $J_q : E \rightarrow 2^{E^*}$  is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|f^*\| \cdot \|x\|, \|f^*\|^q = \|x\|^q\}, \quad \forall x \in E,$$

where  $q > 1$  is a constant. In particular,  $J_2$  is the usual normalized duality mapping. It is known that, in general,  $J_q(x) = \|x\|^2 J_2(x)$ , for all  $x \neq 0$ , and  $J_q$  is single-valued if  $E^*$  is strictly convex.

The modulus of smoothness of  $E$  is the function  $\rho_E : [0, \infty) \rightarrow [0, \infty)$  defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

A Banach space  $E$  is called uniformly smooth if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0.$$

$E$  is called  $q$ -uniformly smooth if there exists a constant  $c > 0$ , such that

$$\rho_E(t) \leq ct^q, \quad q > 1.$$

Note that  $J_q$  is single-valued if  $E$  is uniformly smooth. Xu and Roach [40] proved the following result.

**Lemma 2.1.** *Let  $E$  be a real uniformly smooth Banach space. Then,  $E$  is  $q$ -uniformly smooth if and only if there exists a constant  $c_q > 0$ , such that for all  $x, y \in E$ ,*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + c_q \|y\|^q.$$

**Definition 2.1** (See [41]). Let  $M : \text{dom}(M) \subseteq E \rightarrow 2^E$  be a multi-valued mapping.

(i)  $M$  is said to be accretive if, for any  $x, y \in \text{dom}(M)$ ,  $u \in M(x)$ ,  $v \in M(y)$ , there exists  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle u - v, j_q(x - y) \rangle \geq 0.$$

(ii)  $M$  is said to be  $m$ -accretive if  $M$  is accretive and  $(I + \rho M)(\text{dom}(M)) = E$  holds for every (equivalently, for some)  $\rho > 0$ , where  $I$  is the identity operator on  $E$ .

**Remark 2.1.** It is well known that, if  $E = \mathcal{H}$  is a Hilbert space, then  $M : \text{dom}(M) \subseteq E \rightarrow 2^E$  is  $m$ -accretive if and only if  $M$  is maximal monotone (see, for example, [42]).

Let  $\theta$  be a zero element in  $E$ , and let  $A, B, S, T : E \rightarrow E$  be nonlinear single-valued mappings. Suppose that  $M, N : E \rightarrow 2^E$  are two  $m$ -accretive mappings. We consider the problem of finding  $x^*, y^* \in E$  such that

$$\begin{cases} \theta \in x^* - y^* + \rho(A(y^*) + S(y^*)) + \rho M(x^*) & \text{for } \rho > 0, \\ \theta \in y^* - x^* + \gamma(B(x^*) + T(x^*)) + \gamma N(y^*) & \text{for } \gamma > 0, \end{cases} \quad (2.1)$$

which is called a system of generalized nonlinear mixed quasi-variational inclusions (abbreviated as SGNMQVI) in real Banach spaces.

Below are some special cases of SGNMQVI (2.1).

(i) If  $E = \mathcal{H}$  is a Hilbert space, and  $M, N$  are two maximal monotone mappings, then the SGNMQVI (2.1) reduces to the following system of nonlinear mixed quasi-variational inclusions: find  $x^*, y^* \in \mathcal{H}$  such that

$$\begin{cases} \theta \in \rho(A(y^*) + S(y^*)) + x^* - y^* + \rho M(x^*) & \text{for } \rho > 0, \\ \theta \in \gamma(B(x^*) + T(x^*)) + y^* - x^* + \gamma N(y^*) & \text{for } \gamma > 0. \end{cases} \quad (2.2)$$

This was introduced and studied by Agarwal, Huang and Tan [16].

(ii) If  $E = \mathcal{H}$  is a Hilbert space,  $M = \partial\varphi$ ,  $N = \partial\phi$ , where  $\varphi, \phi : \mathcal{H} \rightarrow R \cup \{+\infty\}$  are two proper convex lower semicontinuous functions on  $\mathcal{H}$  and  $\partial\varphi$  and  $\partial\phi$  denote the subdifferential of functions  $\varphi, \phi$ , respectively, then the SGNMQVI (2.1) reduces to the following generalized system of nonlinear variational inequalities: Find  $x^*, y^* \in \mathcal{H}$  such that

$$\begin{cases} \langle \rho(A(y^*) + S(y^*)) + x^* - y^*, x - x^* \rangle \geq \rho\varphi(x^*) - \rho\varphi(x) & \forall x \in \mathcal{H} \text{ and for } \rho > 0, \\ \langle \gamma(B(x^*) + T(x^*)) + y^* - x^*, x - y^* \rangle \geq \gamma\phi(y^*) - \gamma\phi(x) & \forall x \in \mathcal{H} \text{ and for } \gamma > 0. \end{cases} \quad (2.3)$$

If  $\varphi = \phi$ , then problem (2.3) becomes the problem (1) in [15] which is called the system of generalized nonlinear mixed variational inequalities.

If  $\varphi = \phi$  and  $A = B = 0$ , then problem (2.3) becomes the system of nonlinear mixed variational inequalities in [14] (i.e., problem (2) in [15]).

(iii) If  $\varphi = \phi$  is the indicator of a closed convex subset  $K$  in  $\mathcal{H}$ , that is,

$$\varphi(u) = \phi(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{other,} \end{cases}$$

then the problem (2.3) reduces to the following system of nonlinear variational inequalities: Find  $x^*, y^* \in K$  such that

$$\begin{cases} \langle \rho(A(y^*) + S(y^*)) + x^* - y^*, x - x^* \rangle \geq 0 & \text{for all } x \in K \text{ and for } \rho > 0, \\ \langle \gamma(B(x^*) + T(x^*)) + y^* - x^*, x - y^* \rangle \geq 0 & \text{for all } x \in K \text{ and for } \gamma > 0. \end{cases} \quad (2.4)$$

(iv) If  $A = B = 0$ , then problem (2.1) reduces to the following problem: Find  $x^*, y^* \in E$  such that

$$\begin{cases} \theta \in x^* - y^* + \rho S(y^*) + \rho M(x^*) & \text{for } \rho > 0, \\ \theta \in y^* - x^* + \gamma T(x^*) + \gamma N(y^*) & \text{for } \gamma > 0. \end{cases} \quad (2.5)$$

(v) If  $A = B = 0$ , then problem (2.4) reduces to the following problem: Find  $x^*, y^* \in K$  such that

$$\begin{cases} \langle \rho S(y^*) + x^* - y^*, x - x^* \rangle \geq 0 & \text{for all } x \in K \text{ and for } \rho > 0, \\ \langle \gamma T(x^*) + y^* - x^*, x - y^* \rangle \geq 0 & \text{for all } x \in K \text{ and for } \gamma > 0. \end{cases} \quad (2.6)$$

If  $S = T$ , then problem (2.6) becomes the system of nonlinear variational inequalities introduced and studied by Verma [11,13,17].

We recall some definitions needed later.

**Definition 2.2** (See [41]). Let the multi-valued mapping  $M : \text{dom}(M) \subseteq E \longrightarrow 2^E$  be  $m$ -accretive, for a constant  $\rho > 0$ , the mapping  $R_\rho^M : E \rightarrow \text{dom}(M)$  which is defined by

$$R_\rho^M(u) = (I + \rho M)^{-1}(u), \quad u \in E$$

is called the resolvent operator associated with  $M$  and  $\rho$ .

**Remark 2.2.** It is well known that  $R_\rho^M$  is a single-valued and nonexpansive mapping (see [41]).

**Definition 2.3.** Let  $E$  be a real uniformly smooth Banach space, and  $T : E \rightarrow E$  be a single-valued operator.  $T$  is said to be

(i)  $r$ -strongly accretive if there exists a constant  $r > 0$  such that

$$\langle Tx - Ty, J_q(x - y) \rangle \geq r \|x - y\|^q, \quad \forall x, y \in E,$$

or equivalently,

$$\langle Tx - Ty, J_2(x - y) \rangle \geq r \|x - y\|^2, \quad \forall x, y \in E.$$

(ii)  $s$ -Lipschitz continuous if there exists a constant  $s > 0$  such that

$$\|T(x) - T(y)\| \leq s \|x - y\|, \quad \forall x, y \in E.$$

**Remark 2.3.** If  $T$  is  $r$ -strongly accretive, then  $T$  is  $r$ -expanding, i.e.,

$$\|T(x) - T(y)\| \geq r \|x - y\|, \quad \forall x, y \in E;$$

(iii) If  $E = \mathcal{H}$  is a Hilbert space, then (i) in Definition 2.3 reduces to the  $r$ -strongly monotonicity of  $T$ .

**Lemma 2.2** (See [43]). Let  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  be three real sequences, satisfying:

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 0.$$

where  $t_n \in (0, 1)$ ,  $\sum_{n=0}^{\infty} t_n = \infty$ ,  $\forall n \geq 0$ ,  $b_n = o(t_n)$ ,  $\sum_{n=0}^{\infty} c_n < \infty$ . Then  $a_n \rightarrow 0$ .

**Lemma 2.3.** For given  $x^*, y^* \in E$ ,  $(x^*, y^*)$  is a solution of the SGNMQVI (2.1) if and only if

$$\begin{cases} x^* = R_\rho^M[y^* - \rho(A(y^*) + S(y^*))] & \text{for } \rho > 0, \\ y^* = R_\gamma^N[x^* - \gamma(B(x^*) + T(x^*))] & \text{for } \gamma > 0. \end{cases}$$

**Proof.** By using Definition 2.2, this problem can be proven easily.  $\square$

### 3. Existence and uniqueness

**Theorem 3.1.** Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping,  $A : E \rightarrow E$  be an  $l_1$ -Lipschitz continuous mapping,  $B : E \rightarrow E$  be an  $l_2$ -Lipschitz continuous mapping. If

$$\begin{cases} 0 < (1 - q\rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} + \rho l_1 < 1, \\ 0 < (1 - q\gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} + \gamma l_2 < 1 \end{cases} \quad (3.1)$$

then the problem (2.1) has a unique solution.

**Proof.** First, we prove the existence of a solution. Define a mapping  $F : E \rightarrow E$  as follows:

$$F(x) = R_\rho^M[R_\gamma^N(x - \gamma(B(x) + T(x))) - \rho(A + S)(R_\gamma^N(x - \gamma(B(x) + T(x))))], \quad \forall x \in E.$$

By Lemma 2.2, for all  $x, y \in E$ , we have

$$\begin{aligned} \|F(x) - F(y)\| &= R_\rho^M[R_\gamma^N(x - \gamma(B(x) + T(x))) - \rho(A + S)(R_\gamma^N(x - \gamma(B(x) + T(x))))] \\ &\quad - R_\rho^M[R_\gamma^N(y - \gamma(B(y) + T(y))) - \rho(A + S)(R_\gamma^N(y - \gamma(B(y) + T(y))))] \\ &\leq \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\| \\ &\quad - \rho[S(R_\gamma^N(x - \gamma(B(x) + T(x)))) - S(R_\gamma^N(y - \gamma(B(y) + T(y))))] \\ &\quad + \rho\|A(R_\gamma^N(x - \gamma(B(x) + T(x)))) - A(R_\gamma^N(y - \gamma(B(y) + T(y))))\|. \end{aligned} \quad (3.2)$$

Since  $S : E \rightarrow E$  is  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous, it follows from Lemma 2.1 that

$$\begin{aligned} &\|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\| \\ &\quad - \rho[S(R_\gamma^N(x - \gamma(B(x) + T(x)))) - S(R_\gamma^N(y - \gamma(B(y) + T(y))))] \\ &= \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\|^q \\ &\quad + c_q \rho^q \|S(R_\gamma^N(x - \gamma(B(x) + T(x)))) - S(R_\gamma^N(y - \gamma(B(y) + T(y))))\|^q \\ &\quad - q \langle S(R_\gamma^N(x - \gamma(B(x) + T(x)))) - S(R_\gamma^N(y - \gamma(B(y) + T(y)))) \rangle, J_q(R_\gamma^N(x - \gamma(B(x) + T(x))) \\ &\quad - R_\gamma^N(y - \gamma(B(y) + T(y)))) \\ &\leq \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\|^q \\ &\quad + c_q \rho^q k_1^q \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\|^q \\ &\quad - q \rho s_1 \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\|^q \\ &= (1 - q \rho s_1 + c_q \rho^q k_1^q) \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\|^q. \end{aligned} \quad (3.3)$$

Since  $T : E \rightarrow E$  is  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous,  $A$  is  $l_1$ -Lipschitz continuous and  $B$  is  $l_2$ -Lipschitz continuous, we have

$$\begin{aligned} &\|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\| \\ &\leq \|x - y - \gamma(T(x) - T(y))\| + \gamma\|B(x) - B(y)\| \\ &\leq (\|x - y\|^q - q \gamma \langle T(x) - T(y), J_q(x - y) \rangle + c_q \gamma^q \|T(x) - T(y)\|^q)^{\frac{1}{q}} + \gamma l_2 \|x - y\| \\ &\leq [(1 - q \gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} + \gamma l_2] \|x - y\| \end{aligned} \quad (3.4)$$

$$\begin{aligned} &\|A(R_\gamma^N(x - \gamma(B(x) + T(x)))) - A(R_\gamma^N(y - \gamma(B(y) + T(y))))\| \\ &\leq l_1 \|R_\gamma^N(x - \gamma(B(x) + T(x))) - R_\gamma^N(y - \gamma(B(y) + T(y)))\| \\ &\leq l_1 \|x - y - \gamma(T(x) - T(y)) - \gamma(B(x) - B(y))\| \\ &\leq l_1 \|x - y - \gamma(T(x) - T(y))\| + l_1 \gamma \|B(x) - B(y)\| \\ &\leq l_1 [(1 - q \gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} + \gamma l_2] \|x - y\|. \end{aligned} \quad (3.5)$$

It follows from (3.2)–(3.5) that

$$\|F(x) - F(y)\| \leq \sigma \theta \|x - y\|, \quad \forall x, y \in E, \quad (3.6)$$

where

$$\sigma = (1 - q \rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} + \rho l_1, \theta = (1 - q \gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} + \gamma l_2. \quad (3.7)$$

It follows from (3.1) that  $0 < \sigma < 1$ ,  $0 < \theta < 1$ . Thus, (3.6) implies that  $F$  is a contractive mapping and so, there exists a point  $x^* \in E$  such that  $x^* = F(x^*)$ . Let

$$y^* = R_\gamma^N[x^* - \gamma(B(x^*) + T(x^*))].$$

From the definition of  $F$ , we have

$$\begin{cases} x^* = R_\rho^M[y^* - \rho(A(y^*) + S(y^*))], \\ y^* = R_\gamma^N[x^* - \gamma(B(x^*) + T(x^*))]. \end{cases}$$

By Lemma 2.3, we know that  $(x^*, y^*)$  is a solution of the SGNMQVI (2.1).

Next, we show the uniqueness of the solution. Let  $(u^*, v^*)$  be another solution of the SGNMQVI (2.1). It follows from Lemma 2.3 that

$$\begin{cases} u^* = R_\rho^M[v^* - \rho(A(v^*) + S(v^*))], \\ v^* = R_\gamma^N[u^* - \gamma(B(u^*) + T(u^*))]. \end{cases}$$

As the proof of (3.6), we have

$$\|x^* - u^*\|^2 \leq \sigma\theta\|x^* - u^*\|^2.$$

Since  $0 < \sigma < 1$ ,  $0 < \theta < 1$ , it follows that  $u^* = x^*$  and  $v^* = y^*$ . This completes the proof.  $\square$

If  $E = \mathcal{H}$  is a Hilbert space, then it follows from Remarks 2.1 and 2.3 and Theorem 3.1 that the following theorem holds.

**Theorem 3.2.** Let  $\mathcal{H}$  be a Hilbert space and  $M, N : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximal monotone mappings. Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. If

$$\begin{cases} 0 < \rho < \min \left\{ \frac{2(l_1 - s_1)}{l_1^2 - k_1^2}, \frac{1}{l_1} \right\}, & l_1 < s_1 \\ 0 < \gamma < \min \left\{ \frac{2(l_2 - s_2)}{l_2^2 - k_2^2}, \frac{1}{l_2} \right\}, & l_2 < s_2 \end{cases} \quad (3.8)$$

then the problem (2.2) has a unique solution.

By Theorem 3.2, it is easy to obtain the following corollary.

**Corollary 3.3.** Let  $\mathcal{H}$  be a Hilbert space,  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. If the condition (3.8) holds, then the problem (2.3) has a unique solution.

**Remark 3.1.** Let  $\varphi = \phi$  in Corollary 3.3, we recover Theorem 2.1 in [15]. Hence, Theorems 3.1 and 3.2 extend and improve the corresponding results in [15] and [17] in several aspects.

#### 4. Convergence of iterative algorithms

Before we discuss the approximation solvability of SGNMQVI (2.1) problem and its special cases, we need to introduce some new two-steps iterative algorithms, which contain a number of algorithms as special cases.

**Algorithm 4.1.** For arbitrarily chosen initial point  $x^0 \in E$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$x_{n+1} = (1 - \alpha_n - \lambda_n)x_n + \alpha_n R_\rho^M[y_n - \rho(A(y_n) + S(y_n))] + \lambda_n u_n,$$

$$y_n = (1 - \beta_n - \delta_n)x_n + \beta_n R_\gamma^N[x_n - \gamma(B(x_n) + T(x_n))] + \delta_n v_n,$$

where  $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\delta_n\} \subset [0, 1]$ ,  $\{u_n\}, \{v_n\}$  are bounded sequences in  $E$ , and  $0 \leq \alpha_n + \lambda_n \leq 1$ ,  $0 \leq \beta_n + \delta_n \leq 1$ ,  $\forall n \geq 0$ .

If  $A = B = 0$ , then Algorithm 4.1 reduces to the following algorithm.

**Algorithm 4.2.** For arbitrarily chosen initial point  $x^0 \in E$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n R_\rho^M[y_n - \rho S(y_n)] + \lambda_n u_n, \\y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n R_\gamma^N[x_n - \gamma T(x_n)] + \delta_n v_n,\end{aligned}$$

where  $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\delta_n\} \subset [0, 1], \{u_n\}, \{v_n\}$  are bounded sequences in  $E$ , and  $0 \leq \alpha_n + \lambda_n \leq 1, 0 \leq \beta_n + \delta_n \leq 1, \forall n \geq 0$ .

Let  $\lambda_n = 0, \delta_n = 0$  in Algorithm 4.1, we get the following algorithm.

**Algorithm 4.3.** For arbitrarily chosen initial point  $x^0 \in E$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n R_\rho^M[y_n - \rho(A(y_n) + S(y_n))], \\y_n &= (1 - \beta_n)x_n + \beta_n R_\gamma^N[x_n - \gamma(B(x_n) + T(x_n))],\end{aligned}$$

where  $\{\alpha_n\}, \{\beta_n\} \subset [0, 1], \forall n \geq 0$ .

Let  $\beta_n = 0$  in Algorithm 4.3, we have the following algorithm.

**Algorithm 4.4.** For arbitrarily chosen initial point  $x^0 \in E$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n R_\rho^M[y_n - \rho(A(y_n) + S(y_n))], \\y_n &= R_\gamma^N[x_n - \gamma(B(x_n) + T(x_n))],\end{aligned}$$

where  $\{\alpha_n\} \subset [0, 1], \forall n \geq 0$ .

**Algorithm 4.5.** For arbitrarily chosen initial point  $x^0 \in E$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n R_\rho^M[y_n - \rho(A(y_n) + S(y_n))] + \alpha_n u_n + w_n, \\y_n &= R_\gamma^N[x_n - \gamma(B(x_n) + T(x_n))] + v_n,\end{aligned}$$

where  $\{\alpha_n\} \subset [0, 1], \forall n \geq 0$  and  $\{u_n\}, \{w_n\}, \{v_n\}$  are three sequences in  $E$ .

Let  $M = \partial\varphi, N = \partial\phi$  in Algorithm 4.1, we have

**Algorithm 4.6.** For arbitrarily chosen initial point  $x^0 \in \mathcal{H}$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n R_\rho^{\partial\varphi}[y_n - \rho(A(y_n) + S(y_n))] + \lambda_n u_n, \\y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n R_\gamma^{\partial\phi}[x_n - \gamma(B(x_n) + T(x_n))] + \delta_n v_n,\end{aligned}$$

where  $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\delta_n\}, \{u_n\}, \{v_n\}$  are the same as those in Algorithm 4.1.

**Algorithm 4.7.** For arbitrarily chosen initial point  $x^0 \in K$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n P_K[y_n - \rho(A(y_n) + S(y_n))] + \lambda_n u_n, \\y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n P_K[x_n - \gamma(B(x_n) + T(x_n))] + \delta_n v_n,\end{aligned}$$

where  $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\delta_n\}, \{u_n\}, \{v_n\}$  are the same as those in Algorithm 4.1.

Let  $M = \partial\varphi, N = \partial\phi$  in Algorithm 4.5, we have

**Algorithm 4.8.** For arbitrarily chosen initial point  $x^0 \in \mathcal{H}$ , compute the sequences  $\{x_n\}, \{y_n\}$  such that

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n R_\rho^{\partial\varphi}[y_n - \rho(A(y_n) + S(y_n))] + \alpha_n u_n + w_n, \\y_n &= R_\gamma^{\partial\phi}[x_n - \gamma(B(x_n) + T(x_n))] + v_n,\end{aligned}$$

where  $\{\alpha_n\}, \{u_n\}, \{w_n\}, \{v_n\}$  are the same as those in Algorithm 4.5.

**Remark 4.1.** Both Algorithms 4.1 and 4.5 are iterative processes with errors for SGNMQVI (2.1). However, Algorithms 4.1 and 4.5 are different from each other. Algorithm 4.1 is an iterative process with mean errors, while Algorithm 4.5 is an iterative process with mixed errors.

We now present, based on [Algorithm 4.1](#), the approximation solvability of the SGNMQVI (2.1) problem involving strongly accretive and Lipschitz continuous mappings in a  $q$ -uniformly smooth Banach space setting.

**Theorem 4.1.** *Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping,  $A : E \rightarrow E$  be an  $l_1$ -Lipschitz continuous mapping,  $B : E \rightarrow E$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in E \times E$  is a solution to the SGNMQVI (2.1) problem, the sequences  $\{x_n\}, \{y_n\}$  are generated by [Algorithm 4.1](#) and satisfies:*

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \lambda_n < \infty, \beta_n \rightarrow 1$ .
- (ii)

$$\begin{cases} 0 < (1 - q\rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} + \rho l_1 < 1, \\ 0 < (1 - q\gamma s_2 + c_q \rho^q k_2^q)^{\frac{1}{q}} + \gamma l_2 < 1 \end{cases} \quad (4.1)$$

then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Proof.** Since  $(x^*, y^*) \in E \times E$  is a solution of the SGNMQVI (2.1) problem, it follows from [Lemma 2.3](#) that

$$\begin{cases} x^* = R_{\rho}^M[y^* - \rho(A(y^*) + S(y^*))] & \text{for } \rho > 0, \\ y^* = R_{\gamma}^N[x^* - \gamma(B(x^*) + T(x^*))] & \text{for } \gamma > 0. \end{cases}$$

Let  $L = \sup_{n \geq 0} \{\sup_{n \geq 0} \|u_n - x^*\|, \sup_{n \geq 0} \|v_n - y^*\|, \|x^* - y^*\|\}$ .

Applying [Algorithm 4.1](#), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n - \lambda_n)(x_n - x^*) + \alpha_n(R_{\rho}^M[y_n - \rho(A(y_n) + S(y_n))] - R_{\rho}^M[y^* - \rho(A(y^*) + S(y^*))]) \\ &\quad + \lambda_n(u_n - x^*)\| \\ &\leq (1 - \alpha_n - \lambda_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[(A(y_n) + S(y_n)) - (A(y^*) + S(y^*))]\| + \lambda_n\|u_n - x^*\| \\ &\leq (1 - \alpha_n - \lambda_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[S(y_n) - S(y^*)]\| + \rho\alpha_n\|A(y_n) - A(y^*)\| \\ &\quad + \lambda_n\|u_n - x^*\|. \end{aligned} \quad (4.2)$$

Since  $S$  is  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous, we have by [Lemma 2.1](#) that

$$\begin{aligned} \|y_n - y^* - \rho[S(y_n) - S(y^*)]\|^q &= \|y_n - y^*\|^q - q\rho\langle S(y_n) - S(y^*), J_q(y_n - y^*) \rangle + c_q \rho^q \|S(y_n) - S(y^*)\|^q \\ &\leq \|y_n - y^*\|^q - q\rho s_1 \|y_n - y^*\|^q + c_q \rho^q k_1^q \|y_n - y^*\|^q \\ &= (1 - q\rho s_1 + c_q \rho^q k_1^q) \|y_n - y^*\|^q. \end{aligned} \quad (4.3)$$

Since  $A$  is  $l_1$ -Lipschitz continuous, we have

$$\|A(y_n) - A(y^*)\| \leq l_1 \|y_n - y^*\|. \quad (4.4)$$

From (4.2)–(4.4), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n - \lambda_n)\|x_n - x^*\| + \alpha_n\sigma\|y_n - y^*\| + \rho\alpha_n l_1 \|y_n - y^*\| + \lambda_n\|u_n - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n(\sigma + \rho l_1)\|y_n - y^*\| + \lambda_n L, \end{aligned} \quad (4.5)$$

where  $\sigma = (1 - q\rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} < 1 - \rho l_1$ .

Next, we consider

$$\begin{aligned} \|y_n - y^*\| &= \|(1 - \beta_n - \delta_n)(x_n - y^*) + \beta_n(R_{\gamma}^N[x_n - \gamma(B(x_n) + T(x_n))] - R_{\gamma}^N[x^* - \gamma(B(x^*) + T(x^*))]) \\ &\quad + \delta_n(v_n - y^*)\| \\ &\leq (1 - \beta_n - \delta_n)\|x_n - y^*\| + \beta_n\|x_n - x^* - \gamma(T(x_n) - T(x^*))\| + \gamma\beta_n\|B(x_n) - B(x^*)\| + \delta_n\|v_n - y^*\| \\ &\leq (1 - \beta_n - \delta_n)\|x_n - y^*\| + \beta_n\|x_n - x^* - \gamma(T(x_n) - T(x^*))\| + \gamma l_2 \beta_n \|x_n - x^*\| + \delta_n\|v_n - y^*\|. \end{aligned}$$



Similarly to (4.3), we have

$$\|x_n - x^* - \gamma(T(x_n) - T(x^*))\| \leq \theta \|x_n - x^*\|, \quad (4.6)$$

where  $\theta = (1 - q\gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} < 1 - \gamma l_2$ .

It follows from (4.5) and (4.6), that we have

$$\begin{aligned} \|y_n - y^*\| &\leq (1 - \beta_n - \delta_n) \|x_n - y^*\| + (\beta_n \theta + \gamma l_2 \beta_n) \|x_n - x^*\| + \delta_n L \\ &\leq (1 - \beta_n - \delta_n) \|x_n - x^*\| + \beta_n (\theta + \gamma l_2) \|x_n - x^*\| + (1 - \beta_n - \delta_n) \|x^* - y^*\| + \delta_n L \\ &\leq (1 - \beta_n - \delta_n) \|x_n - x^*\| + \beta_n (\theta + \gamma l_2) \|x_n - x^*\| + (1 - \beta_n - \delta_n) L + \delta_n L \\ &\leq (1 - \delta_n) \|x_n - x^*\| + (1 - \beta_n) L \\ &\leq \|x_n - x^*\| + (1 - \beta_n) L. \end{aligned} \quad (4.7)$$

It follows from (4.5) and (4.7) that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n (\sigma + \rho l_1) [\|x_n - x^*\| + (1 - \beta_n) L] + \lambda_n L \\ &= (1 - \alpha_n [1 - (\sigma + \rho l_1)]) \|x_n - x^*\| + \alpha_n (1 - \beta_n) L + \lambda_n L. \end{aligned} \quad (4.8)$$

Let  $a_n = \|x_n - x^*\|$ ,  $t_n = \alpha_n [1 - (\sigma + \rho l_1)]$ ,  $b_n = \alpha_n (1 - \beta_n) L$ ,  $c_n = \lambda_n L$ .

It is easy to verify that the conditions of Lemma 2.2 are satisfied. Then by (4.8) and Lemma 2.2, we have

$$a_n = \|x_n - x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

And by (4.7), we also obtain that

$$\|y_n - y^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof.  $\square$

Let  $A = B = 0$  in Theorem 4.1, we obtain:

**Theorem 4.2.** Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in E \times E$  is a solution to the problem (2.5), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.2 and satisfies:

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=0}^{\infty} \lambda_n < \infty$ ,  $\beta_n \rightarrow 1$ .
- (ii)

$$\begin{cases} 0 < (1 - q\rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} < 1, \\ 0 < (1 - q\gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} < 1. \end{cases} \quad (4.9)$$

Then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

Let  $\lambda_n = \delta_n = 0$  in Theorem 4.1, and we get the following theorem.

**Theorem 4.3.** Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping,  $A : E \rightarrow E$  be an  $l_1$ -Lipschitz continuous mapping,  $B : E \rightarrow E$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in E \times E$  is a solution to the SGNMQVI (2.1) problem, the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.3 and if  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,  $\beta_n \rightarrow 1$  and the condition (4.1) holds, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

Let  $\beta_n = 1$  in Theorem 4.3, we have

**Theorem 4.4.** Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping,  $A : E \rightarrow E$  be an  $l_1$ -Lipschitz continuous mapping,  $B : E \rightarrow E$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in E \times E$  is a solution to the SGNMQVI (2.1) problem, the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.4. If  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and the condition (4.1) holds, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Remark 4.2.** Let  $E = \mathcal{H}$  be a Hilbert space, by Theorems 4.1–4.4, it is easy to obtain four convergence results of Algorithms 4.1–4.4 for some system of variational inclusions in Hilbert spaces. For example, by Theorem 4.1, we have the following corollary.

**Corollary 4.5.** Let  $\mathcal{H}$  be a Hilbert space, and  $M, N : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximal monotone mappings. Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$  is a solution to the problem (2.2), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.1 and

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \lambda_n < \infty, \beta_n \rightarrow 1$ .
- (ii)

$$\begin{cases} 0 < \rho < \min \left\{ \frac{2(l_1 - s_1)}{l_1^2 - k_1^2}, \frac{1}{l_1} \right\}, & l_1 < s_1, \\ 0 < \gamma < \min \left\{ \frac{2(l_2 - s_2)}{l_2^2 - k_2^2}, \frac{1}{l_2} \right\}, & l_2 < s_2 \end{cases} \quad (4.10)$$

then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

Let  $M = \partial\varphi, N = \partial\phi$  in Corollary 4.5, then we get

**Corollary 4.6.** Let  $\mathcal{H}$  be a Hilbert space,  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping and  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$  is a solution to the problem (2.3), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.6. If  $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \lambda_n < \infty, \beta_n \rightarrow 1$  and the condition (4.10) hold, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

Corollary 4.7 follows from Corollary 4.6.

**Corollary 4.7.** Let  $\mathcal{H}$  be a Hilbert space, and let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in K \times K$  is a solution to the problem (2.4), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.7. If  $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \lambda_n < \infty, \beta_n \rightarrow 1$  and the condition (4.10) hold, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Remark 4.3.** It is easy to see that those results in [11,13] are special cases of Corollary 4.7. Hence, Theorems 4.1–4.4, Corollaries 4.5 and 4.6 generalize and improve those results in [11,13] in several aspects.

We now present, based on Algorithm 4.5, the approximation solvability of the SGNMQVI (2.1) problem in a  $q$ -uniformly smooth Banach space setting.

**Theorem 4.8.** Let  $E$  be a real  $q$ -uniformly smooth Banach space and  $M, N : E \rightarrow 2^E$  be  $m$ -accretive mappings. Let  $S : E \rightarrow E$  be an  $s_1$ -strongly accretive and  $k_1$ -Lipschitz continuous mapping,  $T : E \rightarrow E$  be a  $s_2$ -strongly accretive and  $k_2$ -Lipschitz continuous mapping,  $A : E \rightarrow E$  be an  $l_1$ -Lipschitz continuous mapping,  $B : E \rightarrow E$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in E \times E$  is a solution to the SGNMQVI (2.1) problem, the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.5 and the condition (4.1) holds. If

$$\sum_{n=0}^{\infty} \alpha_n = +\infty, \quad \sum_{n=0}^{\infty} \|w_n\| < +\infty, \quad \lim_{n \rightarrow \infty} \|u_n\| = \lim_{n \rightarrow \infty} \|v_n\| = 0$$

then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Proof.** Since  $x^*, y^* \in E \times E$  is a solution of the SGNMQVI (2.1) problem, it follows from Lemma 2.3 that

$$\begin{cases} x^* = R_\rho^M[y^* - \rho(A(y^*) + S(y^*))] & \text{for } \rho > 0, \\ y^* = R_\gamma^N[x^* - \gamma(B(x^*) + T(x^*))] & \text{for } \gamma > 0. \end{cases}$$

Applying Algorithm 4.5, we have

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(R_\rho^M[y_n - \rho(A(y_n) + S(y_n))] - R_\rho^M[y^* - \rho(A(y^*) + S(y^*))]) + \alpha_n u_n + w_n\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[(A(y_n) + S(y_n)) - (A(y^*) + S(y^*))]\| + \alpha_n\|u_n\| + \|w_n\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[S(y_n) - S(y^*)]\| + \rho\alpha_n l_1\|y_n - y^*\| + \alpha_n\|u_n\| + \|w_n\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\sigma\|y_n - y^*\| + \rho\alpha_n l_1\|y_n - y^*\| + \alpha_n\|u_n\| + \|w_n\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n(\sigma + \rho l_1)\|y_n - y^*\| + \alpha_n\|u_n\| + \|w_n\|, \end{aligned} \quad (4.11)$$

where  $\sigma = (1 - q\rho s_1 + c_q \rho^q k_1^q)^{\frac{1}{q}} < 1 - \rho l_1$ .

Next, we have

$$\begin{aligned} \|y_n - y^*\| &= \|R_\gamma^N[x_n - \gamma(B(x_n) + T(x_n))] + v_n - R_\gamma^N[x^* - \gamma(B(x^*) + T(x^*))]\| \\ &\leq \|x_n - x^* - \gamma(T(x_n) - T(x^*))\| + \gamma\|B(x_n) - B(x^*)\| + \|v_n\| \\ &\leq \theta\|x_n - x^*\| + \gamma l_2\|x_n - x^*\| + \|v_n\|, \\ &= (\theta + \gamma l_2)\|x_n - x^*\| + \|v_n\|, \end{aligned} \quad (4.12)$$

where  $\theta = (1 - q\gamma s_2 + c_q \gamma^q k_2^q)^{\frac{1}{q}} < 1 - \gamma l_2$ .

It follows from (4.11) and (4.12), that we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n(\sigma + \rho l_1)[(\theta + \gamma l_2)\|x_n - x^*\| + \|v_n\|] + \alpha_n\|u_n\| + \|w_n\| \\ &= (1 - \alpha_n[1 - (\sigma + \rho l_1)(\theta + \gamma l_2)])\|x_n - x^*\| + \alpha_n(\sigma + \rho l_1)\|v_n\| + \alpha_n\|u_n\| + \|w_n\|. \end{aligned} \quad (4.13)$$

Let  $a_n = \|x_n - x^*\|$ ,  $t_n = \alpha_n[1 - (\sigma + \rho l_1)(\theta + \gamma l_2)]$ ,  $b_n = \alpha_n(\sigma + \rho l_1)\|v_n\| + \alpha_n\|u_n\|$ ,  $c_n = \|w_n\|$ .

It is easy to verify that the conditions of Lemma 2.2 are satisfied. Then by (4.13) and Lemma 2.2, we have

$$a_n = \|x_n - x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

And by (4.12), we also obtain that

$$\|y_n - y^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof.  $\square$

Let  $E = \mathcal{H}$  be a Hilbert space, by Theorem 4.8, it is easy to obtain the following results.

**Corollary 4.9.** Let  $\mathcal{H}$  be a Hilbert space, and  $M, N : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximal monotone mappings. Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$  is a solution to the problem (2.2), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.5. If  $\sum_{n=0}^{\infty} \alpha_n = +\infty$ ,  $\sum_{n=0}^{\infty} \|w_n\| < +\infty$ ,  $\lim_{n \rightarrow \infty} \|u_n\| = \lim_{n \rightarrow \infty} \|v_n\| = 0$ , and the condition (4.10) hold, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Corollary 4.10.** Let  $\mathcal{H}$  be a Hilbert space, and  $M, N : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be maximal monotone mappings. Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an  $s_1$ -strongly monotone and  $k_1$ -Lipschitz continuous mapping,  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a  $s_2$ -strongly monotone and  $k_2$ -Lipschitz continuous mapping,  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_1$ -Lipschitz continuous mapping,  $B : \mathcal{H} \rightarrow \mathcal{H}$  be an  $l_2$ -Lipschitz continuous mapping. Suppose that  $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$  is a solution to the problem (2.3), the sequences  $\{x_n\}, \{y_n\}$  are generated by Algorithm 4.8. If  $\sum_{n=0}^{\infty} \alpha_n = +\infty$ ,  $\sum_{n=0}^{\infty} \|w_n\| < +\infty$ ,  $\lim_{n \rightarrow \infty} \|u_n\| = \lim_{n \rightarrow \infty} \|v_n\| = 0$ , and the condition (4.10) hold, then the sequences  $\{x_n\}, \{y_n\}$ , respectively, converges strongly to  $x^*, y^*$ .

**Remark 4.4.** Let  $\varphi = \phi$  in Corollary 4.10, we recover Theorem 3.5 in [15]. Hence, Theorem 4.8 and Corollary 4.9 extend and improve Theorems 3.5–3.9 in [15] and Theorem 3.8 in [17] in several aspects.

## Acknowledgement

The authors would like to express their thanks to the referee for helpful suggestions.

## References

- [1] J.S. Pang, Asymmetric variational inequality problems over product sets: Applications and iterative methods, *Math. Program.* 31 (1985) 206–219.
- [2] G. Cohen, F. Chaplais, Nested monotony for variational inequalities over a product of spaces and convergence of iterative algorithms, *J. Optim. Theory Appl.* 59 (1988) 360–390.
- [3] M. Bianchi, Pseudo  $P$ -monotone operators and variational inequalities, Report 6, Istituto di econometria e Matematica per le decisioni economiche, Università Cattolica del Sacro Cuore, Milan, Italy, 1993.
- [4] Q.H. Ansari, J.C. Yao, A fixed point theorem and its applications to a system of variational inequalities, *Bull. Austral. Math. Soc.* 59 (1999) 433–442.
- [5] Q.H. Ansari, S. Schaible, J.C. Yao, Systems of vector equilibrium problems and its applications, *J. Optim. Theory Appl.* 107 (2000) 547–557.
- [6] E. Allevi, A. Gnudi, I.V. Konnov, Generalized vector variational inequalities over product sets, *Nonlinear Anal.* 47 (2001) 573–582.
- [7] G. Kassay, J. Kolumbán, System of multi-valued variational inequalities, *Publ. Math. Debrecen* 54 (1999) 267–279.
- [8] G. Kassay, J. Kolumbán, Z. Páles, Factorization of Minty and Stampacchia variational inequality system, *European J. Oper. Res.* 143 (2002) 377–389.
- [9] J.W. Peng, System of generalized set-valued quasi-variational-like inequalities, *Bull. Austral. Math. Soc.* 68 (2003) 501–515.
- [10] J.W. Peng, X.M. Yang, On existence of a solution for the system of generalized vector quasi-equilibrium problems with upper semicontinuous set-valued maps, *Internat. J. Math. Math. Sci.* 15 (2005) 2409–2420.
- [11] R.U. Verma, Projection methods, algorithms and a new system of nonlinear variational inequalities, *Comput. Math. Appl.* 41 (2001) 1025–1031.
- [12] H. Nie, Z. Liu, K.H. Kim, S.M. Kang, A system of nonlinear variational inequalities involving strongly monotone and pseudocontractive mappings, *Adv. Nonlinear Var. Inequal.* 6 (2) (2003) 91–99.
- [13] R.U. Verma, General convergence analysis for two-step projection methods and application to variational problems, *Appl. Math. Lett.* 18 (2005) 1286–1292.
- [14] R.U. Verma, Iterative algorithms and a new system of nonlinear quasivariational inequalities, *Adv. Nonlinear Var. Inequal.* 4 (1) (2001) 117–124.
- [15] J.K. Kim, D.S. Kim, A new system of generalized nonlinear mixed variational inequalities in Hilbert spaces, *J. Convex Anal.* 11 (1) (2004) 235–243.
- [16] R.P. Agarwal, N.J. Huang, M.Y. Tan, Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions, *Appl. Math. Lett.* 17 (2004) 345–352.
- [17] R.U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, *Math. Sci. Res. Hot-Line* 3 (8) (1999) 65–68.
- [18] A.S. Eman, C.B. Stephen, An iterative method for generalized set-valued nonlinear mixed quasi-variational inequalities, *J. Comput. Appl. Math.* 170 (2004) 423–432.
- [19] M.A. Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, *J. Math. Anal. Appl.* 255 (2001) 589–604.
- [20] N.H. Xiu, J.Z. Zhang, Some recent advances in projection-type methods for variational inequalities, *J. Comput. Appl. Math.* 152 (2003) 559–585.
- [21] J.L. Li, On the existence of solutions of variational-inequalities in Banach spaces, *J. Math. Anal. Appl.* 295 (2004) 115–128.
- [22] S.S. Chang, J.K. Kim, K.H. Kim, On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces, *J. Math. Anal. Appl.* 268 (2002) 89–108.
- [23] M.A. Noor, Modified resolvent splitting algorithms for general mixed variational inequalities, *J. Comput. Appl. Math.* 135 (2001) 111–124.
- [24] J.S. Pang, Asymmetric variational inequality problems over product sets: Applications and iterative methods, *Math. Program.* 31 (1985) 206–219.
- [25] N.J. Huang, A new completely general class of variational inclusions with noncompact valued mappings, *Comput. Math. Appl.* 35 (10) (1998) 9–14.
- [26] R.P. Agarwal, Y.J. Cho, N.J. Huang, Sensitivity analysis for strongly nonlinear quasi-variational inclusions, *Appl. Math. Lett.* 13 (6) (2000) 19–24.
- [27] R.P. Agarwal, N.J. Huang, Y.J. Cho, Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, *J. Inequal. Appl.* 7 (6) (2002) 807–828.
- [28] S. Adly, Perturbed algorithm and sensitivity analysis for a general class of variational inclusions, *J. Math. Anal. Appl.* 201 (1996) 609–630.
- [29] F. Giannessi, Theorems alternative, quadratic programs, and complementarity problems, in: R.W. Cottle, F. Giannessi, J.L. Lions (Eds.), *Variational Inequalities and Complementarity Problems*, John Wiley and Sons, New York, 1980, pp. 151–186.
- [30] X.P. Ding, C.L. Luo, Perturbed proximal point algorithms for generalized quasi-variational-like inclusions, *J. Comput. Appl. Math.* 113 (2000) 153–165.
- [31] J.W. Peng, X.M. Yang, Generalized vector quasi-variational-like inequalities, *J. Inequal. Appl.* 59387 (2006) 1–11.
- [32] X.P. Ding, Existence and algorithms of solutions for nonlinear mixed variational-like inequalities in Banach spaces, *J. Comput. Appl. Math.* 157 (2003) 419–434.

- [33] D.L. Zhu, P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequality problems, *SIAM J. Optim.* 6 (1996) 714–726.
- [34] P. Marcotte, D.L. Zhu, Weak sharp solutions and the finite convergence of algorithms for solving variational inequalities, *SIAM J. Optim.* 9 (1999) 179–189.
- [35] J.P. Crouzeix, P. Marcotte, D.L. Zhu, Conditions ensuring the applicability of cutting-plane methods for solving variational inclusions, *Math. Program. Ser. A* 88 (2000) 521–539.
- [36] J.W. Peng, X.J. Long, Sensitivity analysis for parametric completely generalized strongly nonlinear implicit quasi-variational inclusions, *Comput. Math. Appl.* 50 (5–6) (2005) 869–880.
- [37] J.W. Peng, Set-valued variational inclusions with  $T$ -accretive, *Appl. Math. Lett.* Preprint.
- [38] R.W. Cottle, Complementary and variational problem, *Sympos. Math.* 19 (1976) 177–208.
- [39] P.T. Harker, J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Progr.* 48 (1990) 161–220.
- [40] Z.B. Xu, G.F. Roach, Characteristic inequalities uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.* 157 (1991) 189–210.
- [41] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [42] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1979.
- [43] L.S. Liu, Ishikawa, Mann, Iterative process with error for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* 194 (1995) 114–125.